Question:

A baker makes a cylindrical pie with radius $r = 4.5$ inches and depth $d = 2$ inches. If the baker's spouse eats a slice of pie with a central angle of $\frac{\pi}{3}$ radians, which of the following integrals gives the volume, in cubic inches, of the remaining pie?

A) $\int_0^{\pi/3} 4.5^2 \, d\theta$

B) $\int_0^{5\pi/3} 4.5^2 \, d\theta$

C) $2\int_0^{\pi/3} 4.5^2 \, d\theta$

D) $2\int_0^{5\pi/3} 4.5^2 \, d\theta$

Explanation:

Correct Answer: B

The total volume V of the remaining pie is given by the product of the depth 2 and the surface area S of the remaining pie.

$$V = 2S$$

The surface area of the pie is equal to the area enclosed by the graph of the polar curve $r(\theta)$ between θ_1 and θ_2 and is given by the following formula:

polar area

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta$$

A slice with a central angle of $\frac{\pi}{3}$ is removed, so subtract this from the total radians in a circle 2π to see that the remaining pie has a central angle of $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$. Therefore, the remaining pie goes from $\theta_1 = 0$ to $\theta_2 = \frac{5\pi}{3}$.

Substitute the polar area formula for S in the equation for the volume of the remaining pie, then substitute the above limits of integration and $r = 4.5$ into the resulting expression.

$$2S \quad \text{Volume of remaining pie}$$

$$2\left(\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\,d\theta\right) \quad \text{Substitute polar area formula}$$

$$2\left(\frac{1}{2}\int_{0}^{5\pi/3} 4.5^2\,d\theta\right) \quad \text{Plug in } r = 4.5,\ \theta_1 = 0, \text{ and } \theta_2 = \frac{5\pi}{3}$$

$$\int_{0}^{5\pi/3} 4.5^2\,d\theta \quad \text{Multiply}$$

Therefore, the volume of the remaining pie is given by $\int_{0}^{5\pi/3} \mathbf{4.5^2}\,\boldsymbol{d\theta}$.

(Choice A) $\int_0^{\pi/3} 4.5^2\, d\theta$ may result from mistakenly integrating over the central angle of the slice of pie that was removed, but this gives the volume of the slice and not the remaining pie.

(Choice C) $2\int_0^{\pi/3} 4.5^2\, d\theta$ may result from a combination of the errors described in Choice A and D.

(Choice D) $2\int_0^{5\pi/3} 4.5^2\, d\theta$ may result from not including the factor of $\frac{1}{2}$ in the formula for polar area.

Things to remember:

1. The area enclosed by the graph of the polar curve $r(\theta)$ between θ_1 and θ_2 is given by the following formula:

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} r^2\, d\theta$$

2. Enjoy some pie on Pi Day!